Geometric Subsystem Quantization of the Double Sine–Gordon Kink

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Abstract

We extend the geometric subsystem quantisation programme, initiated for the sine–Gordon kink in [1], to the double sine–Gordon (DSG) equation — a non-integrable scalar field theory with a degenerate double-well potential. The travelling-wave (kink) sector is parametrised by the centre position a and the velocity v. We immerse this moduli space into the classical phase space (the PTSO [2]) of the DSG field; after restriction to a sufficiently small neighbourhood its image is a two-dimensional symplectic submanifold (a local embedding). The pullback of the canonical field-theoretic symplectic form is computed explicitly and yields the canonical Darboux form $dP \wedge da$ with the relativistic momentum $P = Mv/\sqrt{1-v^2}$, where M is the static kink mass. Deformation quantisation via the Moyal product then gives the canonical commutation relation $[a, P] = i\hbar$. The derivation uses only the existence of a finite-energy topological kink and the Lorentz invariance of the theory; it does not require integrability. The physical result reproduces the standard collective-coordinate Hamiltonian treatment of the DSG kink [4, 5], but the method — direct pullback of the field-theoretic symplectic form and deformation quantisation — provides a rigorous geometric foundation for the quantum mechanics of non-integrable solitons within the PTSO framework.

Contents

1 Introduction

The geometric subsystem quantisation programme, initiated in [1] for the sine–Gordon kink, is extended here to a non-integrable model. So far the programme has been carried out for the single kink of the sine–Gordon equation. The present paper takes the natural next step: we apply the same pullback–and–quantize strategy to the double sine–Gordon (DSG) kink, a topological soliton in a field theory that is not integrable. The DSG equation arises in Josephson junction arrays with second harmonic coupling, in spin chains with competing anisotropies, and in the description of ferroelectric domain walls. Its kink–antikink scattering exhibits fractal resonance windows, a hallmark of non-integrability, yet the single-kink sector is as simple as in the sine–Gordon case: it consists of a travelling wave parametrised by position and velocity.

We show that the geometric quantisation of the DSG kink moduli space can be performed *without any use of the inverse scattering transform (IST)* — the symplectic form is pulled back directly from the field-theoretic phase space, using only the explicit kink profile and the Lorentz invariance of the model. The computation mirrors the one for the sine–Gordon kink [1] and yields the symplectic form $\omega = dP \wedge da$ with the relativistic momentum $P = Mv/\sqrt{1-v^2}$, where M is the static kink mass. Deformation quantisation then provides the canonical commutation relation $[a, P] = i\hbar$.

It is important to note that the *physical result* — the canonical symplectic structure on the kink moduli space and the relativistic momentum formula — is not new. A complete Hamiltonian treatment of the DSG kink using collective coordinates was given by Willis, El-Batanouny and Sodano in the mid-1980s [4, 5]. In that approach, two canonical coordinates and their conjugate momenta are *assigned* to describe the centre-of-mass motion and the internal degree of freedom of the kink; the commutation relations are then imposed by fiat. The present work differs in its *method*: instead of postulating the canonical variables, we derive them from first principles by pulling back the exact field-theoretic symplectic form to the moduli space and then applying deformation quantisation via the Moyal product. This provides a rigorous, geometric foundation for the quantum mechanics of the DSG kink and demonstrates that the geometric subsystem programme is not limited to integrable models.

The paper is organised as follows. Section 2 introduces the double sine–Gordon equation and its finite-energy kink solution. Section 3 recalls the definition of the PTSO and the symplectic form of the DSG field. Section 4 defines the kink moduli space and its immersion into the PTSO. Section 5 computes the pullback of the symplectic form in full detail. Section 6 discusses deformation quantisation. Section 7 summarises the results and outlines future directions.

2 The double sine–Gordon equation and its kink

The double sine–Gordon equation in laboratory coordinates (t, x) is

$$\varphi_{tt} - \varphi_{xx} + V'(\varphi) = 0, \qquad V(\varphi) = 1 - \cos\varphi + \frac{\kappa}{2}(1 - \cos 2\varphi), \tag{1}$$

where $\kappa > 0$ is a coupling constant. The potential $V(\varphi)$ has degenerate minima at $\varphi = 0 \pmod{2\pi}$ and is therefore topologically non-trivial: the field can interpolate between the vacuum 0 and the vacuum 2π. When $\kappa = 0$ the model reduces to the ordinary sine–Gordon equation; for $\kappa \neq 0$ the integrability is lost, and kink–antikink scattering becomes chaotic with fractal resonance windows [3].

The static kink $\varphi(t,x) = f_0(x)$ satisfies

$$f_0''(x) = V'(f_0(x)), \qquad \lim_{x\to-\infty} f_0(x) = 0, \quad \lim_{x\to+\infty} f_0(x) = 2\pi. \tag{2}$$

Integrating once gives $\frac{1}{2}(f_0')^2 = V(f_0)$. The profile can be expressed implicitly, but its exact form is not needed for the geometric quantisation. We only require that the kink exists and that its derivative f_0' decays exponentially as $|x| \to \infty$ — this follows from the fact that $V(f_0)$ vanishes at the vacua and f_0 approaches them exponentially fast because the vacua are non-degenerate minima of V.

The static energy (mass) of the kink is

$$M = \int_{-\infty}^{\infty} \big(f_0'(x)\big)^2\,dx, \tag{3}$$

which is finite. For a Lorentz-invariant theory, a moving kink is obtained by a boost:

$$\varphi_{a,v}(t,x) = f_0\big(\gamma(x - vt - a)\big), \qquad \gamma = \frac{1}{\sqrt{1-v^2}}, \quad a \in \mathbb{R},\ v \in (-1,1). \tag{4}$$

The corresponding Cauchy data at time $t = 0$ are

$$\phi_{a,v}(x) = f_0(\gamma(x-a)), \qquad \pi_{a,v}(x) = \partial_t\varphi_{a,v}(0,x) = -\gamma v\, f_0'(\gamma(x-a)). \tag{5}$$

3 The PTSO of the double sine–Gordon equation

The Lagrangian density in (t,x) coordinates is $\mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) - V(\varphi)$. The conjugate momentum is $\pi = \varphi_t$. The total energy

$$E[\varphi,\pi] = \frac{1}{2}\int_{\mathbb{R}} \big(\pi^2 + \varphi_x^2\big)dx + \int_{\mathbb{R}} V(\varphi)\,dx$$

is finite for configurations in the vacuum and kink sectors. By the global well-posedness theorem for semilinear wave equations [6], the Cauchy data at time $t = 0$ uniquely determine the solution, and the set of all finite-energy solutions $\mathcal{S}$ is a smooth Banach manifold modelled on the Sobolev space $\mathcal{X} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$. The canonical symplectic form on $\mathcal{S}$ is

$$\Omega\big((\phi_1,\pi_1),(\phi_2,\pi_2)\big) = \int_{\mathbb{R}} \big(\pi_1\phi_2 - \pi_2\phi_1\big)\,dx, \qquad (\phi_j,\pi_j) \in \mathcal{X}. \tag{6}$$

Definition 3.1 (PTSO of the DSG equation)**.** *The* PDE-specific Time-Shared Object *(PTSO) [2] of the double sine–Gordon equation is the symplectic Banach manifold $(\mathcal{S},\Omega)$ of all finite-energy solutions.*

4 Kink moduli space and its immersion

Definition 4.1 (Kink moduli space)**.** *The kink moduli space is the two-dimensional manifold* $\mathcal{M}_K = \mathbb{R} \times (-1,1)$ *with coordinates* (a,v)*. The map*

$$\Phi : \mathcal{M}_K \longrightarrow \mathcal{S}, \qquad \Phi(a,v) = (\phi_{a,v}, \pi_{a,v}),$$

with the Cauchy data given by (5)*, is called the* kink immersion*.*

Lemma 4.2. Φ *is a smooth injective immersion. After restriction to a sufficiently small neighbourhood of any point, its image is an embedded submanifold of* $\mathcal{S}$*.*

Proof. Smoothness: The static kink profile f_0 is C^∞ because it solves an ODE with a smooth nonlinearity V'. Moreover, f_0' and all higher derivatives decay exponentially as $|x| \to \infty$ (the vacua are non-degenerate). The expressions for $\phi_{a,v}$ and $\pi_{a,v}$ are compositions of the smooth map $(a,v) \mapsto \gamma(x-a)$ with f_0 or f_0', and multiplication by smooth functions of v. For each fixed (a,v), the resulting functions belong to $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively, and the map $(a,v) \mapsto (\phi_{a,v}, \pi_{a,v})$ from $\mathbb{R} \times (-1,1)$ to $\mathcal{X} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$ is Fréchet differentiable, with derivatives given by the pointwise expressions computed below. The exponential decay guarantees that these derivatives lie in $\mathcal{X}$ and that the map is continuous in the $H^1 \times L^2$ topology; iteration yields smoothness.

Injectivity: Suppose $\Phi(a_1,v_1) = \Phi(a_2,v_2)$. From the field component we have $f_0(\gamma_1(x-a_1)) = f_0(\gamma_2(x-a_2))$ for all x. Because f_0 is strictly monotone, it is invertible, so $\gamma_1(x-a_1) = \gamma_2(x-a_2)$. Varying x shows that the two affine functions are identical, hence $\gamma_1 = \gamma_2$ and $\gamma_1 a_1 = \gamma_2 a_2$, which yields $a_1 = a_2$ and $|v_1| = |v_2|$. Now use the momentum component: $-\gamma_1 v_1 f_0'(\gamma_1(x-a_1)) = -\gamma_2 v_2 f_0'(\gamma_2(x-a_2))$. With $\gamma_1 = \gamma_2$ and $a_1 = a_2$, and the fact that f_0' is not identically zero, we obtain $v_1 = v_2$. Hence Φ is injective.

Immersion: We compute the tangent vectors $\partial_a\Phi = (\partial_a\phi, \partial_a\pi)$ and $\partial_v\Phi = (\partial_v\phi, \partial_v\pi)$ as elements of $\mathcal{X}$. They are given explicitly in (7)–(8) below, and belong to $\mathcal{X}$ because f_0', f_0'' decay exponentially. To prove linear independence, suppose

$$c_1\,\partial_a\Phi + c_2\,\partial_v\Phi = 0 \quad \text{in } \mathcal{X}.$$

Since $\mathcal{X}$ embeds continuously into $C(\mathbb{R}) \times C(\mathbb{R})$ (by Sobolev embedding in one dimension), the equality holds pointwise almost everywhere, and by continuity of the functions involved, everywhere. We may evaluate at a convenient point. Choose $x = a$, so that $z = \gamma(x-a) = 0$. At this point the expressions simplify:

$$\begin{aligned}
\partial_a\phi(a) &= -\gamma f_0'(0),\\
\partial_v\phi(a) &= \gamma^3 v\,(a-a) f_0'(0) = 0,\\
\partial_a\pi(a) &= +\gamma^2 v\, f_0''(0),\\
\partial_v\pi(a) &= -\gamma f_0'(0) - \gamma^3 v^2 f_0'(0) - \gamma^4 v^2 (a-a) f_0''(0)\\
&= -\gamma f_0'(0)(1+\gamma^2 v^2).
\end{aligned}$$

Insert these into the linear combination:

$$\begin{aligned}
c_1\,\partial_a\phi(a) + c_2\,\partial_v\phi(a) &= -c_1\gamma f_0'(0) = 0,\\
c_1\,\partial_a\pi(a) + c_2\,\partial_v\pi(a) &= c_1\gamma^2 v f_0''(0) - c_2\gamma f_0'(0)(1+\gamma^2 v^2) = 0.
\end{aligned}$$

Because $f_0'(0) \neq 0$ (the kink is not flat at the centre), the first equation gives $c_1 = 0$. Substituting into the second equation yields $-c_2\gamma f_0'(0)(1+\gamma^2 v^2) = 0$, whence $c_2 = 0$ since $\gamma > 0$, $f_0'(0) \neq 0$, and $1+\gamma^2v^2 > 0$. Thus $c_1 = c_2 = 0$, proving that the differential $d\Phi_{(a,v)}$ is injective at every point. Hence Φ is an immersion.

Finally, an injective immersion from a finite-dimensional manifold into a Banach manifold restricts to an embedding on a sufficiently small neighbourhood of each point; this is a standard consequence of the inverse function theorem for Banach spaces. □

5 Pullback of the symplectic form

We now compute the closed two-form $\omega_K = \Phi^*\Omega$ on $\mathcal{M}_K$.

Proposition 5.1. *$\omega_K = dP \wedge da$, where $P = M\gamma v$ is the physical relativistic momentum of the moving kink.*

Proof. Let $z = \gamma(x-a)$. We need the partial derivatives of the Cauchy data with respect to a and v. We compute them carefully.

Tangents with respect to a:

$$\begin{aligned} \phi_{a,v}(x) &= f_0(z), \qquad z = \gamma(x-a), \\ &\partial_a\phi = f_0'(z)\,\partial_a z = f_0'(z)\,(-\gamma) = -\gamma f_0'(z), \\ \pi_{a,v}(x) &= -\gamma v\, f_0'(z), \\ &\partial_a\pi = -\gamma v\, f_0''(z)\,\partial_a z = -\gamma v\, f_0''(z)\,(-\gamma) = +\gamma^2 v\, f_0''(z). \end{aligned} \tag{7}$$

Tangents with respect to v: Recall $\gamma(v) = 1/\sqrt{1-v^2}$, so $\partial_v\gamma = v\gamma^3$.

$$\begin{aligned} \partial_v z &= \partial_v\big[\gamma(x-a)\big] = (x-a)\,\partial_v\gamma = (x-a)\,v\gamma^3, \\ \partial_v\phi &= f_0'(z)\,\partial_v z = f_0'(z)\,(x-a)v\gamma^3 = \gamma^3 v\,(x-a) f_0'(z), \\ \partial_v\pi &= \partial_v\big[-\gamma v\, f_0'(z)\big] \\ &= -(\partial_v\gamma)\,v\,f_0'(z) - \gamma\,f_0'(z) - \gamma v\,f_0''(z)\,\partial_v z \\ &= -(v\gamma^3)\,v\,f_0'(z) - \gamma f_0'(z) - \gamma v\,f_0''(z)\,(x-a)v\gamma^3 \\ &= -\gamma f_0'(z) - \gamma^3 v^2 f_0'(z) - \gamma^4 v^2 (x-a) f_0''(z). \end{aligned} \tag{8}$$

All functions appearing here are smooth and decay exponentially, because f_0' and f_0'' do. Hence each tangent vector lies in $H^1(\mathbb{R}) \times L^2(\mathbb{R})$, as required.

Now form the pullback of the symplectic form evaluated on the coordinate basis:

$$\omega_K(\partial_a, \partial_v) = \int_{\mathbb{R}} \big(\partial_a\pi\,\partial_v\phi - \partial_v\pi\,\partial_a\phi\big)\,dx.$$

Insert the expressions:

$$\begin{aligned} \partial_a\pi\,\partial_v\phi &= \big(+\gamma^2 v\, f_0''(z)\big)\big(\gamma^3 v\,(x-a) f_0'(z)\big) = \gamma^5 v^2 (x-a)\, f_0'(z) f_0''(z), \\ \partial_v\pi\,\partial_a\phi &= \big[-\gamma f_0'(z) - \gamma^3 v^2 f_0'(z) - \gamma^4 v^2 (x-a) f_0''(z)\big]\big[-\gamma f_0'(z)\big] \\ &= +\gamma^2 (f_0')^2 + \gamma^4 v^2 (f_0')^2 + \gamma^5 v^2 (x-a) f_0'(z) f_0''(z). \end{aligned}$$

Subtracting the two terms, the odd part containing $(x-a)$ cancels exactly:

$$\begin{aligned}\partial_a\pi\,\partial_v\phi-\partial_v\pi\,\partial_a\phi &= \gamma^5v^2(x-a)f_0'f_0''-\big[\gamma^2(f_0')^2+\gamma^4v^2(f_0')^2+\gamma^5v^2(x-a)f_0'f_0''\big]\\ &= -\gamma^2(f_0')^2-\gamma^4v^2(f_0')^2.\end{aligned}$$

Therefore

$$\omega_K(\partial_a,\partial_v)=-\int_{\mathbb{R}}\big(\gamma^2+\gamma^4v^2\big)(f_0')^2\,dx.$$

Change the integration variable to $z=\gamma(x-a)$, so $dx=dz/\gamma$:

$$\omega_K(\partial_a,\partial_v)=-\frac{\gamma^2+\gamma^4v^2}{\gamma}\int_{-\infty}^{\infty}\big(f_0'(z)\big)^2\,dz.$$

Now simplify the factor using the identity

$$\gamma^2+\gamma^4v^2=\gamma^2\big(1+\gamma^2v^2\big)=\gamma^2\Big(1+\frac{v^2}{1-v^2}\Big)=\gamma^2\,\frac{1}{1-v^2}=\gamma^4.$$

Thus

$$\omega_K(\partial_a,\partial_v)=-\frac{\gamma^4}{\gamma}\,M=-M\gamma^3.$$

Finally, we relate this to the physical momentum. Define $P\equiv M\gamma v$, which is the standard relativistic momentum (positive for $v>0$). Then

$$dP=M\,d(\gamma v)=M\big(\gamma\,dv+v\,d\gamma\big)=M\big(\gamma\,dv+v\cdot v\gamma^3\,dv\big)=M\gamma\big(1+v^2\gamma^2\big)dv=M\gamma^3\,dv.$$

Hence

$$\omega_K=-M\gamma^3\,da\wedge dv=-da\wedge dP=dP\wedge da.$$

Thus ω_K is globally Darboux on $\mathcal{M}_K$, with canonical coordinates (a,P) and Poisson bracket $\{a,P\}=1$. □

Remark 5.2 (Sign convention)**.** *The canonical symplectic form on the field phase space is taken as $\Omega=\int(\delta\pi\wedge\delta\phi)\,dx$. With this convention, the pullback to the kink moduli space yields $\omega_K=dP\wedge da$ and the Poisson bracket $\{a,P\}=1$. The same convention is used in the geometric quantization of the sine–Gordon kink [1] and will be maintained throughout the programme.*

6 Deformation quantisation

The Moyal product on $C^\infty(\mathcal{M}_K)[[\hbar]]$,

$$f\star g=f\,\exp\Big(\frac{i\hbar}{2}(\overleftarrow{\partial_a}\overrightarrow{\partial_P}-\overleftarrow{\partial_P}\overrightarrow{\partial_a})\Big)\,g,$$

provides a deformation quantisation of the Poisson bracket $\{a,P\}=1$. In particular, $a\star P-P\star a=i\hbar$. Choosing the Schrödinger polarisation $\partial_P=0$ yields the Hilbert space $L^2(\mathbb{R},da)$ with the Hamiltonian $\hat{H}=\sqrt{-\hbar^2\partial_a^2+M^2}$, the semi-relativistic energy operator of a free quantum particle of mass M.

This completes the geometric subsystem quantisation of the DSG kink.

7 Conclusion and outlook

We have shown that the geometric subsystem quantisation programme extends rigorously to the double sine–Gordon kink, a non-integrable scalar field theory. The key ingredients — a finite-energy topological kink and Lorentz invariance — are sufficient to pull back the field-theoretic symplectic form to the two-dimensional moduli space and to quantise it. No integrability or inverse scattering machinery was used.

The resulting symplectic structure and the commutation relation $[a, P] = i\hbar$ reproduce the well-known collective-coordinate Hamiltonian treatment of the DSG kink [4, 5]. The essential novelty of the present work lies in the *method*: instead of assigning canonical variables by hand, we derived them from the fundamental field-theoretic symplectic form via an explicit pullback, and we applied deformation quantisation (the Moyal product) directly to the resulting finite-dimensional symplectic manifold. This provides a rigorous geometric foundation for the quantum mechanics of the DSG kink and demonstrates that the geometric subsystem programme is not restricted to integrable models.

The symplectic form on the translational moduli space takes the form $dP \wedge da$, with P the relativistic momentum and M the static mass, in both the sine–Gordon [1] and double sine–Gordon cases. This common structure suggests that the programme can be extended to any relativistic scalar field theory with topological kinks. The next natural steps are:

- Extending the pullback to models with several kink species, such as the ϕ^6 model with multiple vacua, where the moduli space becomes a disjoint union of symplectic manifolds.
- Including internal shape modes that are not zero modes but can be treated as additional collective coordinates; this would require a more general geometric framework.
- Quantising multi-kink scattering states in non-integrable models, where asymptotic factorisation still holds and the additive IST formula is replaced by the direct pullback of the symplectic form to the product of single-kink moduli spaces.

These directions are currently under investigation.

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References

[1] A. Timmermans, A. Y. Kalmykov, *Quantization of the Kink Moduli Space in the Sine–Gordon Model and a Conjectural Programme for the General Time-Shared Object*, 2026. Zenodo, `https://doi.org/10.5281/zenodo.20521839`

[2] A. Timmermans, A. Y. Kalmykov, *The TSO—PTSO—GTSO Hierarchy. From ODE-Reduction Classification to a Local PTSO Embedding and a Conjectural Universal Framework*, 2026, Zenodo. `https://doi.org/10.5281/zenodo.20521386`

[3] M. Peyrard, D. K. Campbell, *Kink–antikink interactions in a modified sine-Gordon model*, Physica D **9** (1983), 33–51.

[4] C. R. Willis, M. El-Batanouny, P. Sodano, *Collective Coordinates and Linear Modes of the Double-Sine-Gordon Kink*, in *Competing Interactions and Microstructures: Statics and Dynamics* (Springer Series in Solid-State Sciences, Vol. 69), Springer, 1987.

[5] P. Sodano, M. El-Batanouny, C. R. Willis, *Eigenfunctions of the small oscillations about the double-sine-Gordon kink*, Phys. Rev. B **34** (1986), 4936–4949.

[6] J. Ginibre, G. Velo, *The global Cauchy problem for the non-linear Klein–Gordon equation*, Math. Z. **189** (1987), 487–505.